THE DRIVEN-OSCILLATOR EVOLUTION
IN THE TOMOGRAFIC-PROBABILITY REPRESENTATION

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Abstract

We consider the problem of the driven harmonic oscillator in the probability representation of quantum mechanics, where the oscillator states are described by fair nonnegative probability distributions of position measured in rotated and squeezed reference frames in the systems phase space. For some specific oscillator states like coherent states and nth excited states, the tomographic-probability distributions (called the state tomograms) are found in an explicit form. The evolution equation for the tomograms is discussed for the classical and quantum driven oscillators, and the tomographic propagator for this equation is studied.

Keywords: evolution equation, Moyal equation, Wigner function, symplectic tomogramm, optical tomogramm of quantum state.

1 Introduction

Recently [1–4] the new formulation of quantum mechanics based on identification of any quantum state with the tomographic-probability distribution containing the same information on the state that is given by the state wave function [5] or the density operator [6, 7]was introduced. This formulation is the result of many previous attempts to find such a notion of the quantum state which is similar to the notion of classical state associated with the probability density of the classical system in its phase space (q, p). These attempts provided the notion of the Wigner function $W(q, p)$ introduced in [8], the Husimi function $Q(q, p)$ introduced in [9], as well as Glauber function $P(q, p)$ [10] and Sudarshan function $\varphi(z)$ [11]. All these functions were associated with system density operator by invertable maps. Also all these functions are similar in some aspects to classical probability distribution $f(q, p)$. Nevertheless all these functions, called quasiprobability distributions or quasiprobabilities, are not fair probability distributions in the phase space.

At the same time, optical tomogramm $w(X, \theta)$, introduced as a technical tool [12,13] and related to the Wigner function by the integral Radon transform [14] is a measurable in quantum optics experiments [15] probability distribution of the photon homodyne quadrature X, where the angle θ is the local oscillator phase. Optical tomogramms were used to reconstruct the Wigner function (considered as a concept of

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quantum state) from the experimental data. In [1] the notion of the symplectic tomogramm $w(X, \mu, \nu)$ was suggested. It is also a probability distribution of the photon quadrature X, related with optical tomogramm by an invertible map. Also it became clear that tomographic probabilities can be considered as primary objects (not as technical tools) and can be identified with quantum states. These tomogramms are alternative to the wave function and density operator since they contain complete information about the quantum states.

The problem of driven oscillator is an interesting problem studied first in [16–18]. The coherent states of the oscillator and dynamical invariants [19] of such system were found in [20–22]. On the other hand, a detail consideration of this problem in the tomographic probability representation has never been performed. The aim of our work is to study driven harmonic oscillator in the tomographic probability representation. We obtained the symplectic and optical tomogramms as solutions of the tomographic evolution equation for coherent and excited states of driven oscillator.

This paper is organized as follows.

We find the Green function for the Schrödinger equation in Sec. 2 and the evolution equations for the classical probability distribution in Sec. 3, where we also obtain the Wigner function and classical and quantum tomograms. We study symplectic and optical tomograms of coherent and excited states in Secs. 4 and 5, and finally we give our conclusions in Sec. 6.

2 Green Function of Driven Harmonic Oscillator

We study the problem of a quantum oscillator with a time-dependent external force. The Hamiltonian of such a system reads (we put $m = 1, \omega = 1, \hbar = 1$):

$$\hat{H} = \frac{\hat{q}^2}{2} + \frac{\hat{p}^2}{2} - f(t)\hat{q} \tag{1}$$

Our goal is to find the expression for Green's function $G(x, x', t)$ of Schrödinger evolution equation. For this, we employ the method of integrals of motion [19–22]. We consider a classical analog of the Hamiltonian (1):

$$H(q, p, t) = \frac{q^2}{2} + \frac{p^2}{2} - f(t)q \tag{2}$$

The classical equations of motion determining the oscillator position and momentum read

$$\dot{q} = p \tag{3}$$
$$\dot{p} = -q + f(t) \tag{4}$$

and the classical trajectories in the oscillator phase space under the initial conditions $q(0) = q_0, p(0) = p_0$ are given by formulae:

$$q(t) = q_0 \cos t + p_0 \sin t + \int_0^t \sin(t - \tau)f(\tau)d\tau \tag{5}$$

$$p(t) = p_0 \cos t - q_0 \sin t + \int_0^t \cos(t - \tau)f(\tau)d\tau \tag{6}$$

Now we consider the system of equations (5) and (6) as an algebraic system for unknown initial position q_0 and momentum p_0, respectively. The variables q, p, t are taken as parameters. The solution of the

system is given as:

$$q_0(q, p, t) = q \cos t - p \sin t + \int_0^t \sin \tau f(\tau) d\tau = q \cos t - p \sin t + F(t) \tag{7}$$

$$p_0(q, p, t) = p \cos t + q \sin t - \int_0^t \cos \tau f(\tau) d\tau = p \cos t + q \sin t - J(t) \tag{8}$$

We define operators acting in the Hilbert space of the quantum driven oscillator states as follows:

$$\hat{q}_0(\hat{q}, \hat{p}, t) = \hat{q} \cos t - \hat{p} \sin t + F(t) \tag{9}$$

$$\hat{p}_0(\hat{q}, \hat{p}, t) = \hat{q} \sin t + \hat{p} \cos t - J(t) \tag{10}$$

Calculating the total derivative of the operator $\hat{q}_0(q, p, t)$ with respect to time t we obtain:

$$\frac{d\hat{q}_0}{dt} = \frac{\partial \hat{q}_0}{\partial t} + i\left[\hat{H}, \hat{q}_0\right] = \dot{F}(t) - f(t) \sin t = 0 \tag{11}$$

where we take into account the definition of function $F(t) = \int_0^t \sin \tau f(\tau) d\tau$ given by the (7).

Similarly, the total time derivative of the operator $\hat{p}_0(q, p, t)$:

$$\frac{d\hat{p}_0}{dt} = \frac{\partial \hat{p}_0}{\partial t} + i\left[\hat{H}, \hat{p}_0\right] = -\dot{J}(t) + f(t) \cos t = 0 \tag{12}$$

where the definition of function $J(t) = \int_0^t \cos \tau f(\tau) d\tau$ from (8) is taken into account.

Thus, operators (9) and (10) are integrals of motion and correspond to initial position and momentum. Then these operators must satisfy the equations for the Green's function $G(x, x', t)$ [20–22]:

$$\hat{q}_{0(x)} G(x, x', t) = \hat{q}_{(x')} G(x, x', t) \tag{13}$$

$$\hat{p}_{0(x)} G(x, x', t) = -\hat{p}_{(x')} G(x, x', t) \tag{14}$$

where the operators in the left-hand sides of equations act on variable x, and in the right-hand sides on x'. Now we write Eqs. (13) and (14) explicitly:

$$\left(x \cos t + i\frac{\partial}{\partial x} \sin t + F(t)\right) G(x, x', t) = x' G(x, x', t) \tag{15}$$

$$\left(x \sin t - i\frac{\partial}{\partial x} \cos t - J(t)\right) G(x, x', t) = i\frac{\partial}{\partial x'} G(x, x', t) \tag{16}$$

where we still need to define function $\tilde{x}(t)$ to obtain more convenient equations,

$$F(t) \cos t - J(t) \sin t = -\int_0^t \sin(t - \tau) f(\tau) d\tau = -\tilde{x}(t) \tag{17}$$

It is worth noting that $\tilde{x}(t)$ is solution of the classical equations of motion (3), (4) with zero initial conditions: $q(0) = 0, p(0) = 0$.

Finally, system of equations for defining Green's function $G(x, x', t)$ is:

$$\frac{\partial G}{\partial x} = \left(-\frac{x \cos t - x'}{i \sin t} - \frac{F(t)}{i \sin t}\right) G \tag{18}$$

$$\frac{\partial G}{\partial x'} = \left(-\frac{x' \cos t - x}{i \sin t} - \frac{\tilde{x}(t)}{i \sin t}\right) G \tag{19}$$

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where we skipped the arguments of the Green function.

Now one can integrate (18) with respect to variable x and (19) with respect to variable x'. Afterperforming the integration, we arrive at

$$G(x, x', t) = C(t) \exp\left(-\frac{(x^2 + x'^2)\cos t - 2xx'}{2i \sin t} - \frac{xF(t) + x'\tilde{x}(t)}{i \sin t}\right) \tag{20}$$

Finally, following the procedure adopted from [19–22], we obtain the equation for the function $C(t)$ as follows:

$$i\frac{\partial G(x, x', t)}{\partial t} = \hat{H}_{(x)}G(x, x', t) \tag{21}$$

After some algebra, we obtain an equation that does not contain the variables x, x' is obtained:

$$i\frac{dC(t)}{dt} = C(t)\left(\frac{\text{ctg}\, t}{2i} + \frac{F^2(t)}{2\sin^2 t}\right) \tag{22}$$

Equation (22) can be simply integrated with respect to time t and one obtains the function $C(t)$ in a form containig a constant C:

$$C(t) = \frac{C}{\sqrt{\sin t}} \exp\left(-\frac{i}{2}\int_0^t \frac{F^2(\tau)}{\sin^2 \tau} d\tau\right) \tag{23}$$

To calculate the constant C we require that the following condition at the initial moment of time $t = 0$ be satisfied:

$$G(x, x', t = 0) = \delta(x - x') \tag{24}$$

After calculating C using (24) and substituting it into (23) and (20) we arrive at the final expression for the Green function of the Schrödinger evolution equation for the wave function,

$$G(x, x', t) = \frac{1}{\sqrt{2\pi i \sin t}} \exp\left(-\frac{i}{2}\int_0^t \frac{F^2(\tau)}{\sin^2 \tau} d\tau\right) \times$$
$$\times \exp\left(-\frac{(x^2 + x'^2)\cos t - 2xx'}{2i \sin t} - \frac{xF(t) + x'\tilde{x}(t)}{i \sin t}\right) \tag{25}$$

We consider some particular cases as examples.

We start with the case of a constant external force, $f(t) = f_0 = $ const. In this particular case we calculate functions $\tilde{x}(t)$ and $F(t)$ according to their definitions (17) and (7) correspondingly and substitute the result into the final expression (25).After some algebra, we obtain

$$G(x, x', t) = \frac{1}{\sqrt{2\pi i \sin t}} \exp\left(\frac{if_0^2 t}{2} - \frac{\left((x - f_0)^2 + (x' - f_0)^2\right)\cos t - 2(x - f_0)(x' - f_0)}{2i \sin t}\right) \tag{26}$$

Taking into account that Hamiltonian in this particular case differs from Hamiltonian of a harmonic oscillator with shifted equilibrium just in a constant term:

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + (\hat{q} - f_0)^2\right) - \frac{f_0^2}{2}$$

we write the following set of equalities:

$$\hat{H}_{(x)}G(x,x',t) = \exp\left(\frac{if_0^2 t}{2}\right) i\frac{\partial}{\partial t}\left(G(x,x',t)\exp\left(-\frac{if_0^2 t}{2}\right)\right) - \frac{f_0^2}{2}G(x,x',t) = i\frac{\partial}{\partial t}G(x,x',t)$$

This means that Green function satisfies the Schrödinger evolution equation.

Now we considerthe limit case $\omega \to 0$.

First of all we should recover $m, \omega, \hbar$ in formulae according to the following rules:

$$t \to \omega t; x \to \sqrt{\frac{m\omega}{\hbar}}x; f \to \frac{f}{\omega\sqrt{m\hbar\omega}}$$

After all the substitutions we have the Green function in the form

$$G(x,x',t) =$$
$$= \sqrt{\frac{m}{2\pi i\hbar}}\sqrt{\frac{\omega}{\sin\omega t}}\exp\left[\frac{i}{2\omega^3 m\hbar}\left(f_0^2\omega t + 2f_0^2\frac{\cos\omega t - 1}{\sin\omega t} - 2f_0 m\omega^2\left(x+x'\right)\frac{\cos\omega t - 1}{\sin\omega t} + \right. \right.$$
$$\left. \left. + m^2\omega^4\frac{\left(x^2+x'^2\right)\cos\omega t - 2xx'}{\sin\omega t}\right)\right] \quad (27)$$

At $\omega \to 0$ the Green function reads:

$$G(x,x',t) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left[\frac{i}{\hbar}\left(-\frac{f_0^2}{m}\frac{t^3}{24} + \frac{f_0 t\left(x+x'\right)}{2} + \frac{m\left(x-x'\right)^2}{2t}\right)\right] \quad (28)$$

By direct substitution into the Schrödinger evolution equationwe can check the correctness of our calculations. It is also worth noting that the Green-function phase is the classical action $S(x,x',t)$, which satisfies the Hamilton-Jacobi equation, multiplied by $\frac{i}{\hbar}$.

3   Time Evolution of Classical and Quantum Distribution Functions

Now we consider the propagators of different distribution functions of the system with Hamiltonian (1) in the quantum case and Hamiltonian (2)in the classical case. We start from the classical distribution function and the Liouville equation,

$$\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial q} + (f(t) - q)\frac{\partial}{\partial p}\right)\rho(q,p,t) = 0 \quad (29)$$

Noting that the system trajectories in the phase space are given by Eqs. (5), (6) (and, correspondingly, integrals of motion are $q_0(q,p,t), p_0(q,p,t)$ from (7), (8)), we conclude that the evolution of the function $\rho(q,p,t)$ reads:

$$\rho(q,p,t) = \rho_0(q\cos t - p\sin t + F(t), q\sin t + p\cos t - J(t)) \quad (30)$$

where $\rho_0(q,p)$ is the distribution function at initial moment of time $t=0$ and it depends on two variables q, p only.

Now we consider the Moyal's equation [23] for the Wigner's function $W(q,p,t)$:

$$\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial q} - \frac{1}{i}\left[U\left(q \to q + \frac{i}{2}\frac{\partial}{\partial p}\right) - c.c.\right]\right)W(q,p,t) = 0 \quad (31)$$

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where the quantities in the square brackets are pure imaginary and in the equations, pure real ones. The term in the square brackets can be rewritten, in view of the fact that the potential energy is given by $U(x) = \dfrac{x^2}{2} - f(t)x$, as follows:

$$U\left(q + \frac{i}{2}\frac{\partial}{\partial p}\right) = \Re U + \frac{i}{2}\frac{\partial}{\partial p}\left(q - f(t)\right) \tag{32}$$

We are interested just in imaginary part of this expression. After substituting this into (31) we obtain the equation:

$$\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial q} + (f(t) - q)\frac{\partial}{\partial p}\right)W(q, p, t) = 0 \tag{33}$$

It is obvious that Eq. (33), being the evolution equation of the Wigner function $W(q, p, t)$ fully coincides with the equation (29) of evolution of the classical distribution function.The dependence of the Wigner function on time is given by an expression similar to (30),

$$W(q, p, t) = W_0(q\cos t - p\sin t + F(t), q\sin t + p\cos t - J(t)) \tag{34}$$

Now we move to the tomographic-probability-distribution functions.

We start from classical tomogramm distribution. In thisl case we have [4, 24] the following evolution equation:

$$\left(\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial \nu} + \nu F\left(q \rightarrow -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu}, t\right)\frac{\partial}{\partial X}\right)w(X, \mu, \nu, t) = 0 \tag{35}$$

where $F(q, t)$ has the meaning of acting force and in our case it is given by the expression $F(q, t) = -q + f(t)$. So, evolution equation in this case has following form:

$$\left(\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial \nu} + \nu\frac{\partial}{\partial \mu} + \nu f(t)\frac{\partial}{\partial X}\right)w(X, \mu, \nu, t) = 0 \tag{36}$$

The integrals of the equation (36) can be easily obtained. They read:

$$\nu_0(X, \mu, \nu, t) = \nu\cos t + \mu\sin t \tag{37}$$
$$\mu_0(X, \mu, \nu, t) = \mu\cos t - \nu\sin t \tag{38}$$
$$X_0(X, \mu, \nu, t) = X - \mu\tilde{x}(t) - \nu\tilde{p}(t) \tag{39}$$

where functions $\tilde{x}(t), \tilde{p}(t)$ are solutions of classical equations of motion (3), (4) with zero initial conditions. One should note that at the initial moment of time $t = 0$:

$$\nu_0 = \nu, \mu_0 = \mu, X_0 = X$$

hence, evolution of the tomographic-probability distribution is given by:

$$w(X, \mu, \nu, t) = w_0\left(X - \mu\tilde{x}(t) - \nu\tilde{p}(t), \mu\cos t - \nu\sin t, \nu\cos t + \mu\sin t\right) \tag{40}$$

where $w_0(X, \mu, \nu)$ is the tomographic-probability distribution at the initial moment of time and it depends just on three variables X, μ, ν.

We move now to the quantum tomogram and write the evolution equation according to [1, 25]:

$$\left(\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial \nu} - i\left[U\left(q \rightarrow -\left(\frac{\partial}{\partial X}\right)^{-1}\frac{\partial}{\partial \mu} - i\frac{\nu}{2}\frac{\partial}{\partial X}, t\right) - c.c.\right]\right)w(X, \mu, \nu, t) = 0 \tag{41}$$

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We are interested just in the imaginary part of the expression in the square brackets. It reads $\nu\dfrac{\partial}{\partial\mu} + \nu f(t)\dfrac{\partial}{\partial X}$, and, finally, the evolution equation is

$$\left(\frac{\partial}{\partial t} - \mu\frac{\partial}{\partial\nu} + \nu\frac{\partial}{\partial\mu} + \nu f(t)\frac{\partial}{\partial X}\right) w(X, \mu, \nu, t) = 0 \tag{42}$$

that fully coincides with the equation in the classical case (36). Therefore, evolution of quantum tomogramm is given by the same equation (40).

4 Symplectic Tomograms of Coherent and Excited States

Now we study two examples of evolution of quantum tomogramm distributions.

The first example is the coherent-state evolution, and the second example is the evolution of the nth excited state of the harmonic oscillator.

The wave function of the harmonic-oscillator coherent state [10, 11, 26] at the initial moment of time can be written as follows:

$$\psi(x) = \frac{1}{\sqrt[4]{\pi}}\exp\left(-\frac{(x-x_0)^2}{2} + ip_0 x\right) \tag{43}$$

We calculate the tomographic-probability distribution of the harmonic-oscillator coherent state at the initial moment of time using the following formula [24]:

$$w(X, \mu, \nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(y)\exp\left(\frac{i\mu}{2\nu}y^2 - \frac{iX}{\nu}y\right)dy\right|^2 \tag{44}$$

After performing the integration, we arrive at

$$w_0(X, \mu, \nu) = \frac{1}{\sqrt{\pi}\sqrt{\mu^2+\nu^2}}\exp\left(-\frac{(X-\nu p_0 - \mu x_0)^2}{\nu^2 + \mu^2}\right) \tag{45}$$

Now the time dependence of the tomographic-probability distribution is given, in view of (40), as follows::

$$w(X, \mu, \nu, t) = \frac{1}{\sqrt{\pi}\sqrt{\mu^2+\nu^2}}\exp\left(-\frac{(X-\mu x_{cl}(t) - \nu p_{cl}(t))^2}{\nu^2 + \mu^2}\right) \tag{46}$$

where the functions $x_{cl}(t), p_{cl}(t)$ are solutions of classical equations of motion (3), (4) under the initial conditions $x(0) = x_0, p(0) = p_0$.

Now we consider the evolution of the nth excited state of the harmonic oscillator with the wave function

$$\psi_n(x) = \frac{1}{\sqrt{2^n n!\sqrt{\pi}}}e^{-x^2/2}H_n(x) \tag{47}$$

where $H_n(x) = (-1)^n e^{x^2}\dfrac{d^n}{dx^n}e^{-x^2}$ is Hermite polynomial. We show two ways of finding the evolution of the system — using the propagator for the wave function (Green function $G(x, x', t)$ from (25)) and the

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evolution of the quantum tomogram given by (40).

To study the evolution of the wave function, we start from the integral

$$I_n = \int \exp\left[\left(-\frac{1}{2} + \alpha\right)x^2 + \beta x\right] H_n(x) dx \tag{48}$$

where α is a pure imaginary quantity. As a result, we obtained [27]:

$$I_n = \sqrt{\frac{2\pi}{1 - 2\alpha}} \exp\left(\frac{\beta^2}{2(1 - 2\alpha)}\right) i^n \left(\frac{1 + 2\alpha}{\sqrt{1 - 4\alpha^2}}\right)^n H_n\left(-\frac{i\beta}{\sqrt{1 - 4\alpha^2}}\right) \tag{49}$$

and for the evolution of the wave function

$$\psi(x, t) = \int G(x, x't)\psi(x', t = 0)dx' = \frac{e^{-i(n+1/2)t}}{\sqrt{2^n n! \sqrt{\pi}}} \times$$
$$\times \exp\left(-\frac{i}{2}\left(\int_0^t \frac{F^2(\tau)}{\sin^2 \tau} d\tau + \tilde{x}^2(t)\operatorname{ctg} t\right)\right) \exp\left(-\frac{(x - \tilde{x}(t))^2}{2} + i\tilde{p}(t)x\right) H_n(x - \tilde{x}(t)) \tag{50}$$

Now we use formula (44) to obtain the expression for the tomographic-probability distribution at any particular moment of time t from expression for the wave function (50):

$$w_n(X, \mu, \nu, t) = \frac{1}{2^n n! \sqrt{\pi}} \frac{1}{\sqrt{\nu^2 + \mu^2}} \exp\left(-\frac{(X - \mu\tilde{x}(t) - \nu\tilde{p}(t))^2}{\nu^2 + \mu^2}\right) H_n^2\left(\frac{X - \mu\tilde{x}(t) - \nu\tilde{p}(t)}{\sqrt{\nu^2 + \mu^2}}\right) \tag{51}$$

Finally, we show the second way of calculations.

The tomographic-probability distribution at the initial moment of time $t = 0$ according to formula (44) reads:

$$w_n(X, \mu, \nu, t = 0) = \frac{1}{2^n n! \sqrt{\pi}} \frac{1}{\sqrt{\nu^2 + \mu^2}} \exp\left(-\frac{X^2}{\nu^2 + \mu^2}\right) H_n^2\left(\frac{X}{\sqrt{\nu^2 + \mu^2}}\right) \tag{52}$$

For the nth excited state of the harmonic oscillator, the mean values of the position and momentum are equal to zero, so the initial conditions for the functions $x_{cl}(t), p_{cl}(t)$ are zero and these functions coincide with $\tilde{x}(t), \tilde{p}(t)$. Then using formula for the evolution of the tomographic-probability distribution (40) we obtain the same time dependence of the tomogramm distribution (51).

5 Optical Tomograms of the Driven Oscillator

In this section, we present all the results obtained in the previous sections in terms of the optical tomogram $w(X, \theta)$. The evolution equation for the optical tomogram can be derived from the Moyal

equation (31) using folowing correspondence rules [28]:

$$qW(q,p) \longleftrightarrow \left(\left(\frac{\partial}{\partial X} \right)^{-1} \sin\theta \frac{\partial}{\partial\theta} + X\cos\theta \right) w(X,\theta) \tag{53}$$

$$pW(q,p) \longleftrightarrow \left(-\left(\frac{\partial}{\partial X} \right)^{-1} \cos\theta \frac{\partial}{\partial\theta} + X\sin\theta \right) w(X,\theta) \tag{54}$$

$$\frac{\partial}{\partial q} W(q,p) \longleftrightarrow \cos\theta \frac{\partial}{\partial X} w(X,\theta) \tag{55}$$

$$\frac{\partial}{\partial p} W(q,p) \longleftrightarrow \sin\theta \frac{\partial}{\partial X} w(X,\theta) \tag{56}$$

After such substitutionswe can write Eq. (33) in the form containing the optical tomogram $w(X,\theta,t)$:

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial\theta} + f(t)\frac{\partial}{\partial X}\sin\theta \right) w(X,\theta,t) = 0 \tag{57}$$

Integrals of motion for Eq. (57) can be written in a form, similar to formulae (37)-(39):

$$\theta_0(X,\theta,t) = \theta + t \tag{58}$$

$$X_0(X,\theta,t) = X - \tilde{x}(t)\cos\theta - \tilde{p}(t)\sin\theta \tag{59}$$

In view of (58) and (59) we obtain the final expression for the time evolution of the quantum optical tomogram of the driven harmonic oscillator

$$w(X,\theta,t) = w_0(X - \tilde{x}(t)\cos\theta - \tilde{p}(t)\sin\theta, \theta + t) \tag{60}$$

where $w_0(X,\theta)$ is optical tomogramm distribution at the initial moment of time $t = 0$ and it depends on just two variables X,θ.

Optical tomograms at the initial moment of time $w_0(X,\theta)$ for the cases considered in previous section can be found from (45), (52) using the substitution $\mu \longrightarrow \cos\theta, \nu \longrightarrow \sin\theta$. So, for the coherent state of the driven oscillator the time evolution of the optical tomogram reads

$$w(X,\theta,t) = \frac{1}{\sqrt{\pi}} \exp\left(-(X - x_{cl}(t)\cos\theta - p_{cl}(t)\sin\theta)^2 \right) \tag{61}$$

In the case of the nth excited state of the harmonic oscillator, the evolution of the optical tomogram is

$$w_n(X,\theta,t) = \frac{1}{2^n n!\sqrt{\pi}} \exp\left(-(X - \tilde{x}(t)\cos\theta - \tilde{p}(t)\sin\theta)^2 \right) H_n^2 \left(X - \tilde{x}(t)\cos\theta - \tilde{p}(t)\sin\theta \right) \tag{62}$$

6   Conclusions

To conclude, we point out the main results of our study.

We presented a short review of the tomographic probability representation of quantum mechanics on the example of the driven harmonic oscillator. We considered the evolution equation for the tomographicprobability densities of the symplectic and optical tomograms. We concentrated on the important solutions of the evolution equations which correspond to the Gaussian packets (normal distributions of the coherent state) and GaussHermite distributions corresponding to the oscillator excited states. Other tomographic approaches to the driven-oscillator problem like the photon-number tomographic scheme will be studied in future publications.

References

[1] S. Mancini, V. I. Manko, and P. Tombesi, *Phys. Lett. A*, **213**, 1 (1996).

[2] A. Ibort, V. I.Manko, G. Marmo, A. Simoni, and F. Ventriglia, *Physica Scripta* **79**, 065013 (2009).

[3] M. A. Man'ko and V. I. Man'ko, *Found. Phys.* **41**, 330 (2011)

[4] O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **18**, 407 (1997).

[5] E. Schrödinger, *Annalen d. Physik*, **79**, 361 (1926); **81**, 109 (1926).

[6] L. D. Landau, *Ztschr. Physik*, **45**, 430 (1927).

[7] J. von Neumann, *Göttingen. Nachr.*, 245 (1927).

[8] E. P. Wigner, *Phys. Rev.*, **40**, 749 (1932).

[9] K. Husimi, *Proc. Phys. Math. Soc. Jpn.*, **22**, 264 (1940).

[10] R. J. Glauber *Phys. Rev. Lett.*, **10**, 84 (1963).

[11] E. C. G. Sudarshan, *Phys. Rev. Lett.*, **10**, 177 (1963).

[12] J. Bertrand and P. Bertrand, *Found. Phys.*, **17**, 397 (1987).

[13] K. Vogel and H. Risken, *Phys. Rev. A*, **40**, 2847 (1989).

[14] J. Radon, *Ber. Verh. Sachs. Akad.*, **69**, 262 (1917).

[15] D. T. Smithey, M. Beck, M. G. Raymer and A. Faridani, *Phys. Rev. Lett.*, **70**, 1244 (1993).

[16] K. Husimi, *Progr. Theor. Phys.*, **9**, 381 (1953).

[17] J. Schwinger, *Phys. Rev.*, **91**, 728 (1953).

[18] R. P. Feynmann, *Phys. Rev.* , **80**, 440 (1950).

[19] S. K. Suslov, *Physica Scripta*, **81**, 055006(2010); see also arXiv:1002.0144v6 [math-ph] 11 Mar 2010.

[20] I. A. Malkin, V. I. Man'ko, and D. A. Trifonov, *Phys. Rev. D*, **2**, 1371 (1970).

[21] I. A. Malkin and V. I. Man'ko, *Dynamic Symmetries and Coherent States of Quantum Systems* [in Russian], Nauka, Moscow, 1979.

[22] V. V. Dodonov and V. I. Man'ko, *invariants and the Evolution of Nonstationary Quantum Systems. Proceedings of Lebedev Physical Institute*, Nauka, Moscow, 1987, Vol. 183 [translated by Nova Science, N. Y., 1989]

[23] J. E. Moyal, *Proc. Cambridge Phil. Soc.*, **45**, 99 (1949).

[24] V. I. Man'ko and R. V. Mendes, *Phys. Lett. A*, **263**, 53 (1999).

[25] S. Mancini, V. I. Manko, and P. Tombesi, *Found. Phys.*, **27**, 801 (1997).

[26] J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics*, W. A. Benjamin Inc., N. Y., 1968.

[27] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 2007.

[28] Y. A. Korennoy and V. I. Manko, *J. Russ. Laser Res.*, **32**, 74 (2011).